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SAIN Approximation in $C[a, b]$

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The SAIN approximation scheme introduced by Deutsch and Morris [1–3] may be stated as follows:

SAIN APPROXIMATION SCHEME. *Suppose M is a dense subset of a normed linear space X and $\{x_1^*,..., x_n^*\}$ is a finite subset of the dual X^* of X. Given $x \in X$, approximate it by an $m \in M$ for which $x_i^*m = x_i^*x$ ($i = 1,..., n$) and $\| m \| = \| x \|$.*

The SAIN approximation problem is to determine what n-tuples of linear functionals $x_1^*,..., x_n^*$ will be such that any $x \in X$ may be approximated arbitrarily closely by an $m \in M$ under the SAIN approximation scheme. Equivalently, for what n-tuples of linear functionals does a Weierstrass theorem hold for the SAIN approximation scheme?

Several authors have contributed to solving the SAIN approximation problem, both in abstract and concrete spaces (e.g., [5, 6, 8–10, 12]). We consider the SAIN approximation problem in the concrete space of all continuous functions on a compact interval, where we take the dense subset M of $C[a, b]$ to be the polynomials Π. The result obtained can be generalized to some other dense subspaces M of $C[a, b]$, some cases of which will be given below.

Even though the proof below is more complex, the characterization obtained for the solution of the SAIN approximation problem is as simple as that of the related OSAS approximation problem dealt with by the author [7] earlier. The compendium of the results below is stated in Theorem 2, located at the end of Section 3.

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1. Preliminaries

In their fundamental paper [2], Deutsch and Morris observed that one obtained an affirmative answer to the SAIN approximation problem on $C(T)$, T compact Hausdorff, whenever M was a dense subalgebra of $C(T)$ and the linear functional side conditions were all point evaluations. We rephrase their result as Proposition A below. Though most recent work on the SAIN approximation problem has not dealt directly with function spaces, one exception is an interesting result obtained by Lambert [9] which we give as Proposition C.

The goal of this paper is to study the SAIN approximation problem for the special case of T being a compact interval and M the polynomials. The characterizations obtained are useful and easy to apply in concrete problems.

Lemma A [11, 4]. *If x^* is a bounded linear functional on $C(T)$, T compact Hausdorff, then there exist positive linear functionals u^*, v^* on $C(T)$ such that*

$$x^* = u^* - v^*, \qquad \| x^* \| = \| u^* \| + \| v^* \|.$$

Furthermore the u^*, v^* are uniquely defined by the x^*.

Lemma B [11, 4]. *If x^* is a [positive] linear functional on $C(T)$, then there exists a finite [positive] Borel measure μ such that*

$$u^*(f) = \int f \, d\mu \qquad (f \in C(T)).$$

We recall that by the support of a bounded linear functional x^* we mean the support of the finite Borel measure μ representing x^*.

Definition 1. We say that a linear functional x^* has *finitely atomic* support (is *purely finitely atomic*) in case the associated Borel measure is (i) purely atomic, and (ii) has at most a finite number of atoms.

Proposition A [2]. *Suppose M is a dense subalgebra of $C(T)$, T compact Hausdorff. If $x_1^*, \ldots, x_n^*$ each have finitely atomic support, then given $f \in C(T)$ and $\epsilon > 0$ arbitrary there is an $m \in M$ such that $x_i^* m = x_i^* f$ $(i = 1, \ldots, n)$, $\| m \| = \| f \|$, and $\| f - m \| < \epsilon$.*

Lemma C [4]. *Suppose X is a normed linear space, $\{c_1, \ldots, c_n\}$ arbitrary scalars, $\{x_1^*, \ldots, x_n^*\}$ a finite subset of the dual X^*, and $\lambda > 0$. Then for any $\epsilon > 0$, there exists an $x \in X$ such that*

$$x_i^* x = c_i \quad (i = 1, \ldots, n) \qquad \textit{and} \qquad \| x \| < \lambda + \epsilon, \tag{1}$$

if and only if

$$\left|\sum \alpha_i c_i\right| \leqslant \lambda \left\|\sum \alpha_i x_i^*\right\| \tag{2}$$

for all $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$.

At times a slightly stronger result is desired than Lemma C, which we can get using Yamabe's theorem [13]:

LEMMA D (Yamabe's theorem). *Suppose M is a dense convex subset of a normed linear space X. For $x_1^*, ..., x_n^* \in X^*$, $\epsilon > 0$, and $x \in X$ there is an $m \in M$ such that $x_i^* m = x_i^* x$ $(i = 1, ..., n)$ and $\| x - m \| < \epsilon$.*

LEMMA E. *If M is a dense subspace of X, under the hypotheses of Lemma C there exists an $m \in M$ such that*

$$x_i^* m = c_i \quad (i = 1, ..., n) \qquad \text{and} \qquad \| m \| < \lambda + \epsilon \tag{3}$$

if and only if (2) *holds.*

PROPOSITION B [6]. *Suppose M is a dense subspace of a normed linear space X. If $x_1^*, ..., x_n^* \in X^*$ and $x \in X \backslash M$ are such that there exists an $m \in M$ such that*

$$x_i^* m = x_i^* x \quad (i = 1, ..., n) \qquad \text{and} \qquad \| m \| < \| x \|$$

then given $\epsilon > 0$ arbitrary there is an $r \in M$ such that

$$x_i^* r = x_i^* x \qquad (i = 1, ..., n),$$
$$\| r \| < \| x \| \qquad \text{and} \qquad \| x - r \| < \epsilon.$$

PROPOSITION C [9]. *Suppose $f \in C(T)$, T compact Hausdorff. If f attains its norm at most finitely often on T, then given any linear functionals $x_1^*, ..., x_n^*$ on $C(T)$, any $\epsilon > 0$ and any dense subalgebra M of $C(T)$, there exists an $m \in M$ such that*

$$\begin{gathered} x_i^* m = x_i^* f \qquad (i = 1, ..., n), \\ \| m \| = \| f \| \qquad \text{and} \qquad \| f - m \| < \epsilon. \end{gathered} \tag{4}$$

Current terminology is to say that the triple $(C(T), M, \{x_1^*, ..., x_n^*\})$ has property SAIN if and only if the conclusion (4) above holds for some $m = m(\epsilon) \in M$, for any $\epsilon > 0$ and $f \in C(T)$ arbitrary.

DEFINITION 2. Suppose X is a normed linear space, and M a dense subset of X. A linear functional $x^* \in X^*$ is said to be a *SAIN functional* in case the triple (X, M, x^*) has property SAIN. A finite sequence $x_1^*,..., x_n^*$ is said to be a *SAIN sequence* in case every $x \in \langle x_1^*,..., x_n^*\rangle$ is a SAIN functional, where $\langle x_1^*,..., x_n^*\rangle$ is the linear span of $x_1^*,..., x_n^*$.

Remark 1. A necessary condition that a triple $(X, M, \{x_1^*,..., x_n^*\})$ have property SAIN holding is that the sequence $x_1^*,..., x_n^*$ be a SAIN sequence. We investigate the converse of this statement below.

We will use the notation $\mathscr{P}$ to denote the cone of positive linear functionals defined on a function space. supp x^* will designate the support of the functional x^*. We will also designate the $u^*[v^*]$ of Lemma A by x^{+*} or $(x^*)^+$ [x^{-*} or $(x^*)^-$] and call it the positive (resp. negative) part of x^*. χ_B will denote the characteristic function of the subset B of $[a, b]$. The norm used in Euclidean space $\mathbb{R}^\nu$ will be l_1-norm:

$$\|(\alpha_1,..., \alpha_\nu)\| = \sum_{j=1}^{\nu} |\alpha_j|.$$

DEFINITION 3. If $x \in X$ and $x^* \in X^*$, x^* is said to be *nonextremal with respect to* x in case $|x^*x| < \|x^*\| \|x\|$. A finite sequence $x_1^*,..., x_n^*$ of linear functionals is said to be *nonextremal with respect to* x in case every nonzero $x^* \in \langle x_1^*,..., x_n^*\rangle$ is nonextremal with respect to x. A sequence $x_1^*,..., x_n^*$ is said to be *nonextremal* in case it is nonextremal with respect to every nonzero $x \in X$.

LEMMA 1. *If $x_1^*,..., x_n^*$ are linearly independent linear functionals nonextremal with respect to an $x \in X$, then given $\epsilon > 0$ arbitrary there is an $m \in M$ such that*

$$x_i^*m = x_i^*x \qquad (i = 1,..., n),$$
$$\|m\| < \|x\| \qquad \textit{and} \qquad \|x - m\| < \epsilon,$$

whenever M is a dense subspace of the normed linear space X.

Proof. Let $S = \{x^* \in \langle x_1^*,..., x_n^*\rangle; \|x^*\| = 1\}$. Then the expression

$$|x^*f|/\|x^*\| \|f\| \tag{5}$$

is a continuous function of $x^* \in S$, and stricted bounded above by 1 for every $x^* \in S$. But S is compact, so (5) must attain its supremum. Hence there is a $0 < \lambda < 1$ such that $|x^*f| \leq \lambda \|x^*\| \|f\|$ holds for all $x^* \in \langle x_1^*,..., x_n^*\rangle \backslash \{0\}$. The conclusion now follows from Lemma E and Proposition B. ∎

COROLLARY 1. *If M is a dense subspace of a normad linear space X and $x_1^*,..., x_n^*$ is a nonextremal SAIN sequence on X, then the triple $(X, M, \{x_1^*,..., x_n^*\})$ has property SAIN.*

We recall that to show property SAIN holds for a triple $(X, M, \{x_1^*, \ldots, x_n^*\})$, it suffices [2] to show that given $\epsilon > 0$ and $f \in X$ arbitrary that there exists a $p \in M$ for which $x_i^* p = x_i^* f$ $(i = 1, \ldots, n)$, $\| p \| \leqslant \| f \|$ and $\| f - p \| < \epsilon$. We use $x^* \circ \chi_A$ to denote the restriction of x^* to $C[A]$, i.e., $(x^* \circ \chi_A)(f) = x^*(\chi_A f)$ for any $f \in C[a, b]$.

2. $C[a, b]$ AND Π: SUFFICIENCY

LEMMA 2. *If $[a, b]$ is a compact interval and x^* a bounded linear functional on $C[a, b]$, then x^* is a SAIN functional (with respect to $X = C[a, b]$ and $M = \Pi$) if and only if either*

(i) *x^* has finitely atomic support, or*

(ii) *$x^* \in \pm\mathscr{P}$ and* $\operatorname{supp} x^* = [a, b]$, *or*

(iii) $\operatorname{supp} x^{+*} \cap \operatorname{supp} x^{-*} \neq \varnothing$.

Proof. If neither (i), (ii), nor (iii) hold, necessarily $x^* = x^{+*} - x^{-*}$ with $\operatorname{supp} x^{+*}$ disjoint from $\operatorname{supp} x^{-*}$. By Urysohn's lemma we may construct a continuous function g on $[a, b]$ so that $g(x) = 1$ on $\operatorname{supp} x^{+*}$, -1 on $\operatorname{supp} x^{-*}$, and $|g(x)| \leqslant 1$ otherwise. If $p \in \Pi$ is such that $x^* p = x^* g$, necessarily $x^{+*} p = x^{+*} g$ and $x^{-*} p = x^{-*} g$. At most one of x^{+*}, x^{-*} may be purely finitely atomic and neither has support all of $[a, b]$. Suppose that x^{+*} is neither purely finitely atomic nor the zero linear functional on $C[a, b]$. But then any $p \in \Pi$ such that $x^{+*} p = x^{+*} g = \| x^{+*} \|$ must be one on a set of positive measure, whence necessarily identically one on $[a, b]$. If $\operatorname{supp} x^{-*} \neq \varnothing$, $\| g - p \| \geqslant 2$ and done. Thus we may suppose $\operatorname{supp} x^{-*} = \varnothing$. Since $\operatorname{supp} x^{+*} \neq [a, b]$, let $t \in [a, b] \backslash \operatorname{supp} x^{+*}$ and define a continuous $h \in C[a, b]$ so that $h = 1$ on $\operatorname{supp} x^{+*}$, -1 at t, and so that $|h(x)| \leqslant 1$ elsewhere on $[a, b]$. Then as above any $p \in \Pi$ such that $x^{+*} p = x^{+*} h = \| x^{+*} \|$ must be one on a set of positive measure, and so $\| p - h \| \geqslant 2$.

Conversely, (i) is a special case of Proposition A, while (ii) and (iii) are a special case of Proposition B, for if (ii), $|x^* f| = \| x^* \|$ if and only if $f = \mathbb{1} \in \Pi$, and if (iii), x^* does not attain its norm on $C[a, b]$. ∎

LEMMA 3. *Suppose $x_1^*, \ldots, x_n^*$ is a SAIN sequence in $C[a, b]$. Then there are at most finitely many $t \in [a, b]$ such that e_t, $x_1^*, \ldots, x_n^*$ is not a SAIN sequence in $C[a, b]$, where e_t is point evaluation at t.*

Proof. By induction. By Lemma 2, an x^* is not a SAIN functional if and only if $\operatorname{supp} x^* \neq [a, b]$, $\operatorname{supp} x^{+*} \cap \operatorname{supp} x^{-*} = \varnothing$, and x^* does not have purely finitely atomic support. Suppose $n = 1$. If $x^* \in \langle e_t, x_1^* \rangle$ is not a SAIN functional, necessarily $x^* = e_t + \xi x_1^*$ for some $\xi \in \mathbb{R}$. If $\operatorname{supp} x_1^* =$

$[a, b]$, so is supp x^*, and if x_1^* is purely finitely atomic so is x^*. Hence suppose that supp $x_1^{+*} \cap$ supp $x_1^{-*} \neq \varnothing$. In order for x^* not to be a SAIN functional it is necessary that supp $x^{+*} \cap$ supp $x^{-*} = \varnothing$, whence clearly we must have supp $x_1^{+*} \cap$ supp $x_1^{-*} = \{t\}$, a singleton set. Thus if $n = 1$, at most one $t \in [a, b]$ may exist so that e_t, x_1^* is not a SAIN sequence.

Suppose valid for $n = k$. In order that an $x^* \in \langle e_t, x_1^*, ..., x_{k+1}^*\rangle$ not to be a SAIN linear functional, except for finitely many $t \in [a, b]$, necessarily $x^* = \xi_0 e_t + \xi_1 x_1^* + \cdots + \xi_{k+1} x_{k+1}^*$, no $\xi_j = 0$. Since $x_1^*, ..., x_{k+1}^*$ is a SAIN sequence, in order for x^* not to be a SAIN functional it is necessary that $\{t\} = \operatorname{supp}(\xi_1 x_1^* + \cdots + \xi_{k+1} x_{k+1}^*)^+ \cap \operatorname{supp}(\xi_1 x_1^* + \cdots + \xi_{k+1} x_{k+1}^*)^-$ and that t is an atom of $\xi_1 x_1^* + \cdots + \xi_{k+1} x_{k+1}^*$. We therefore consider functionals of the form $x_{k+1}^* + y^*$, $y^* \in \langle x_1^*, ..., x_k^*\rangle$. Suppose $t_1, t_2 ....$ in $[a, b]$ and $y_1^*, y_2^*, ...$ in $\langle x_1^*, ..., x_k^*\rangle$ are such that $\operatorname{supp}(x_{k+1}^* + y_1^*)^+ \cap \operatorname{supp}(x_{k+1}^* \cap y_1^*)^- = \{t_1\}$, $\operatorname{supp}(x_{k+1}^* + y_2^*)^+ \cap \operatorname{supp}(x_{k+1}^* + y_2^*)^- = \{t_2\}, ...,$ and that moreover t_1 is an atom of $x_{k+1}^* + y_1^*$, t_2 is an atom of $x_{k+1}^* + y_2^*$, etc. Since $\dim\langle x_1^*, ..., x_k^*\rangle = k < +\infty$, at most k of the $y_1^*, y_2^*, ...$ are linearly independent; suppose $y_1^*, ..., y_k^*$ are. Suppose $y_{k+1}^* = \alpha_1 y_1^* + \cdots + \alpha_k y_k^*$. At least one of the coefficients α_i is not zero; suppose $\alpha_1 \neq 0$. Since $x_{k+1}^* + y_{k+1}^* = x_{k+1}^* = \alpha_1 y_1^* + \alpha_2 y_2^* + \cdots + \alpha_k y_k^*$ is such that $\operatorname{supp}(x_{k+1}^* + y_{k+1}^*)^+ \cap \operatorname{supp}(x_{k+1}^* + y_{k+1}^*)^- = \{t_{k+1}\}$, either t_{k+1} is some $t_1, ..., t_k$ or else none of the $t_1, ..., t_k$ is an atom of $x_{k+1}^* + \alpha_1 y_1^* + \cdots + \alpha_k y_k^*$. But each t_j is an atom of $x_{k+1}^* + y_j^*$, whence necessarily each t_j is also an atom of $\alpha_1 y_1^* + \cdots + (\alpha_j - 1) y_j^* + \cdots + \alpha_k y_k^*$, and in fact necessarily of $\alpha_1 y_1^* + \cdots + \alpha_{j-1} y_{j-1}^* + \alpha_{j+1} y_{j+1}^* + \cdots + \alpha_k y_k^*$, and hence of at least one of the y_i^*, $i \neq j$, which has a nonzero coefficient α_i. On the other hand t_{k+1} is an atom of $x_{k+1}^* + \alpha_1 y_1^* + \cdots + \alpha_k y_k^*$, so t_{k+1} is an atom of either x_{k+1}^* or some y_i^*. If t_{k+1} is an atom of x_{k+1}^*, then t_{k+1} not being an atom of $x_{k+1}^* + y_j^*$ for every $j = 1, ..., k$ implies that t_{k+1} has to be an atom of every y_j^*. Hence t_{k+1} is an atom of some y_i^*. By the pigeon-hole principle, two of the $t_1, ..., t_{k+1}$ have to be atoms of the same y_i^*, for some $i = 1, ..., k$. Suppose t_k, t_{k+1} are atoms of y_1^*. Then $x_{k+1}^* + y_1^*$ not having t_{k+1} as an atom implies t_{k+1} must be an atom of x_{k+1}^*, and hence of every y_i^*, $i = 1, ..., k$. Similarly, for every $\mu \geqslant k + 1$, t_μ must be an atom of x_{k+1}^*, and hence of every y_i^*, $i = 1, ..., k$. Thus, we may decompose x^* as

$$x^* = \sum_{\mu=k+1}^{\infty} \psi_\mu e_{t_\mu} + w^*, \quad \psi_\mu \neq 0 \qquad \text{for all} \quad \mu \geqslant k + 1,$$

where w^* does not have any of the $t_{k+1}, t_{k+2}, ...$ as atoms. Similarly, we must have, for each $i = 1, ..., k + 1$,

$$y_i^* = \sum_{\mu=k+1}^{\infty} \rho_{i,\mu} e_{t_\mu} + z_i^*, \quad \rho_{i,\mu} \neq 0 \qquad \text{for all } \mu,$$

and in fact necessarily $\rho_{i,\mu} = \psi_\mu$ for every μ and i, where none of the $t_{k+1}, t_{k+2},\ldots$ is an atom of z_i^*. But then

$$y_{k+1}^* = \sum_{i=1}^{k} \alpha_i y_i^*$$

$$= \sum_{\mu=k+1}^{\infty} \sum_{i=1}^{k} \alpha_i \psi_\mu e_{t_\mu} + \sum_{i=1}^{k} \alpha_i z_i^*,$$

and if we set $z^* = \sum_{i=1}^{k} \alpha_i z_i^*$, none of the $t_{k+1}, t_{k+2},\ldots$ can be an atom of z^*. But then each of the $t_{k+2}, t_{k+3},\ldots$ is also an atom of y_{k+1}^*, while not of $x_{k+1}^* + y_{k+1}^*$, so necessarily $\sum_{i=1}^{k} \alpha_i \psi_\mu$, being the coefficient of t_μ in y_{k+1}^*, must be the negative of the coefficient of t_μ in x_{k+1}^*, for each $\mu \geqslant k+2$, whence necessarily $\sum_{i=1}^{k} \alpha_i = -1$. But then t_{k+1} is not an atom of $x_{k+1}^* + y_{k+1}^*$ either, a contradiction. Thus there can be at most finitely many values of $t \in [a, b]$ such that $\operatorname{supp}(x_{k+1}^* + y^*)^+ \cap \operatorname{supp}(x_{k+1}^* + y^*)^- = \{t\}$, y^* being in $\langle x_1^*,\ldots, x_{k+1}^*\rangle$. ∎

COROLLARY 2. *Suppose* $x_1^*,\ldots, x_n^*$ *is a SAIN sequence in* $C[a, b]$, *and* $a \leqslant t_1 < t_2 < \cdots < t_u \leqslant b$. *If* $a < t_1$ *and* $t_u < b$, *there exist sequences* $x_{i,\eta}, y_{i,\eta}$ *such that*

$$\begin{aligned}
&\text{(i)} \quad a < x_{1,\eta} < t_1 < y_{1,\eta} < x_{2,\eta} < t_2 < y_{2,\eta} < \cdots \\
&\qquad < x_{u,\eta} < t_u < y_{u,\eta} < b, \\
&\text{(ii)} \quad x_{i,\eta} \nearrow t_i \quad (i = 1,\ldots, n), \\
&\text{(iii)} \quad y_{i,\eta} \searrow t_i \quad (i = 1,\ldots, n), \qquad\qquad (6)\\
&\text{(iv)} \quad | t_i - x_{i,n} | = | t_i - y_{i,\eta} | = \eta \ (i = 1,\ldots, n), \text{ and} \\
&\text{(v)} \quad e_{x_{1,\eta}}, e_{y_{1,\eta}},\ldots, e_{x_{u,\eta}}, e_{y_{u,\eta}}, x_1^*,\ldots, x_n^* \text{ is a SAIN sequence on } C[a, b].
\end{aligned}$$

If $a = t_1$, (6) *is valid with the sequence* $x_{1,\eta}$ *deleted; if* $t_u = b$, (6) *is valid with the sequence* $y_{u,\eta}$ *deleted.*

LEMMA 4. *Suppose* $x_1^*,\ldots, x_n^*$ *are linearly independent linear functionals on* $C[a, b]$. *Then at most finitely many* $t \in [a, b]$ *exist so that* $e_t, x_1^*,\ldots, x_n^*$ *are not linearly independent on* $C[a, b]$.

Proof. Suppose not, and let $t_1, t_2,\ldots$ in $[a, b]$, $\xi_{i,j} \in \mathbb{R}$ be such that $e_{t_j} = \sum_{i=1}^{n} \xi_{i,j} x_i^*$. Let $\xi_j = (\xi_{1,j}, \xi_{2,j},\ldots, \xi_{n,j}) \in \mathbb{R}^n$. Since the $e_{t_1}, e_{t_2},\ldots$ are linearly independent on $C[a, b]$, necessarily the $\xi_j \in \mathbb{R}^n$ are also. But there can be at most n linearly independent $\xi_j \in \mathbb{R}^n$, a contradiction. ∎

LEMMA 5. *Suppose* $x_1^*,\ldots, x_n^*$ *are linearly independent linear functionals on* $C[a, b]$. *Suppose* $m \in \Pi$, $\| m \| < 1$, *and let* $c_i = x_i^* m$ $(i = 1,\ldots, n)$. *Then* *there exists a* $\tau > 0$ *such that* $d \in \mathbb{R}^n$, $\| d - c \| < \tau$ *implies there exists an* $r \in \Pi$ *for which both* $\| r \| < 1$ *and* $x_i^* r = d_i$.

Proof. By Lemma C there exists an $\eta < 1$ for which

$$\left| \sum \alpha_i c_i \right| \leqslant \eta \left\| \sum \alpha_i x_i^* \right\| \qquad (\alpha \in \mathbb{R}^n).$$

By hypothesis, $\| \sum \alpha_i x_i^* \| = 0$ if and only if $\alpha = 0$. Hence, if $\sigma = \min\{\| \sum \alpha_i x_i^* \|; \alpha \in \mathbb{R}^n, \| \alpha \| = 1\}$, and $\tau < (1 - \eta)\sigma/n$, then if $d \in \mathbb{R}^n$ is such that $\| c - d \| < \tau$,

$$\begin{aligned} \left| \sum \alpha_i d_i \right| &\leqslant \left| \sum \alpha_i c_i \right| + \sum | \alpha_i | \, | c_i - d_i | \\ &< \eta \left\| \sum \alpha_i x_i^* \right\| + (1 - \eta)\sigma \\ &\leqslant \left\| \sum \alpha_i x_i^* \right\|. \end{aligned}$$

But the continuous function

$$\left| \sum \alpha_i d_i \right| \Big/ \left\| \sum \alpha_i x_i^* \right\|$$

attains its norm on the compact set $\{\alpha \in \mathbb{R}^n; \| \alpha \| = 1\}$, whence there is an $\eta' < 1$ such that

$$\left| \sum \alpha_i d_i \right| \leqslant \eta' \left\| \sum \alpha_i x_i^* \right\| \qquad (\alpha \in \mathbb{R}^n).$$

By Lemma E there is then an $r \in \Pi$ such that $x_i^* r = d_i$ $(i = 1,..., n)$ and $\| r \| < 1$. ∎

COROLLARY 3. *Suppose* $e_{t_1}, ..., e_{t_u}$, $x_1^*, ..., x_n^*$ *are linearly independent on* $C[a, b]$. *Then there exists a positive constant* τ *such that given any* $d \in \mathbb{R}^n$ *having norm less than* $\tau\sigma$ *there is an* $m \in \Pi$ *for which*

(i) $m(t_j) = 0$ $(j = 1,..., u)$,
(ii) $x_i^* m = d_i$ $(i = 1,..., n)$, *and* (7)
(iii) $\| m \| < \sigma$.

Proof. By Lemma 5 there is an $s \in \Pi$ for which $s(t_j) = 0$ $(j = 1,..., u)$, $x_i^* s = d_i$ $(i = 1,..., n)$, and $\| s \| < 1$ whenever $\| d \| < \tau$, $d \in R^n$ (take the zero polynomial for m in the hypotheses of Lemma 5). Hence, for $\sigma > 0$,

$m = \sigma s$ satisfies (7) for the data σd, $\| d \| < \tau$. But $\{\sigma d; \| d \| < \tau\} = \{d; \| d \| < \tau\sigma\}$. Since $\| m \| = \sigma \| s \| < \sigma$, the conclusion follows. ∎

LEMMA 6. *Suppose e_t, $x_1^*,..., x_n^*$ is a SAIN sequence on $C[a, b]$. Let $w_i^* = x_i^* \circ \chi_{[a,b]\setminus\{t\}} = x_i^* - x_i^* \circ \chi_{\{t\}}$, $i = 1,..., n$. Then $w_1^*,..., w_n^*$ is also a SAIN sequence on $C[a, b]$.*

Proof. Each w_i^* is a linear combination of e_t and x_i^*, whence $\langle w_1^*,..., w_n^*\rangle \subset \langle e_t, x_1^*,..., x_n^*\rangle$. By hypothesis any $x^* \in \langle e_t, x_1^*,..., x_n^*\rangle\setminus\{0\}$ is a SAIN functional; hence any $x^* \in \langle w_1^*,..., w_n^*\rangle\setminus\{0\}$ must also be a SAIN functional, whence $w_1^*,..., w_n^*$ is a SAIN sequence on $C[a, b]$. ∎

COROLLARY 4. *Suppose $e_{t_1},..., e_{t_u}$, $x_1^*,..., x_n^*$ is a SAIN sequence on $C[a, b]$. If $w_i^* = x_i^* \circ \chi_{[a,b]\setminus\{t_1,...,t_u\}}$, then $w_1^*,..., w_n^*$ is also a SAIN sequence on $C[a, b]$.*

LEMMA 7. *Suppose $e_{t_1},..., e_{t_u}$, $x_1^*,..., x_n^*$ is a SAIN sequence on $C[a, b]$. Suppose $x_1^*,..., x_n^*$ is nonextremal on $C[a, b]\setminus\Pi$. If $w_i^* = x_i^* \circ \chi_{[a,b]\setminus\{t_1,...,t_u\}}$, then $w_1^*,..., w_n^*$ is also nonextremal.*

Proof. If not, suppose $f \in C[a, b]\setminus\Pi$, $\|f\| = 1$ and $w^* \in \langle w_1^*,..., w_n^*\rangle$, $\| w^* \| = 1$, are such that $| w^*f | = 1$. By Corollary 4, w^* is a SAIN functional on $C[a, b]$. Since $f \neq \pm\mathbb{1}$, w^* cannot be a positive or negative linear functional having support $[a, b]$. Since $| w^*f | = \| w^* \|$, necessarily $\operatorname{supp} w^{+*} \cap \operatorname{supp} w^{-*} = \varnothing$. By Lemma 2, necessarily w^* is finitely purely atomic, whence the $x^* \in \langle x_1^*,..., x_n^*\rangle$ such that $w^* = x^* \circ \chi_{[a,b]\setminus\{t_1,...,t_u\}}$ must also have finitely purely atomic support. But then x^* is not nonextremal for $C[a, b]\setminus\Pi$, a contradiction. ∎

LEMMA 8. *Suppose $x_1^*,..., x_n^*$ are linearly independent linear functionals on $C[a, b]$, and that t_i, $x_{i,\eta}$, $y_{i,\eta}$ $(i = 1,..., u)$ are sequences of points in $[a, b]$ such that $x_{i,\eta} \nearrow t_i$, $y_{i,\eta} \searrow t_i$ as $\eta \to 0^+$ $(i = 1,..., u)$. Then either $e_{t_1},..., e_{t_u}$, $x_1^*,..., x_n^*$ are linearly dependent or else*

$$\liminf_{\eta\to 0^+} \min_{\substack{\alpha\in R^{n+2u} \\ \|\alpha\|=1}} \left\| \sum_{i=1}^{n} \alpha_i x_i^* + \sum_{i=n+1}^{n+u} \alpha_i e_{x_{i,\eta}} + \sum_{i=n+u+1}^{n+2u} \alpha_i e_{y_{i,\eta}} \right\| \tag{8}$$

is strictly positive.

Proof. By Lemma 4, at most finitely many points $t \in [a, b]$ exist for which e_t, $x_1^*,..., x_n^*$ are linearly dependent. Let B_1 be the set of these points. Then, for $t_1 \notin B_1$, at most finitely many points $t \in [a, b]$ exist for which e_t, e_{t_1}, $x_1^*,..., x_n^*$ are linearly dependent. Let B_2 be this set. In this manner we obtain finite sets $B_1, B_2, \ldots, B_u$ for which $e_{t_1}, \ldots, e_{t_u}, x_1^*, \ldots, x_n^*$ are linearly independent whenever $t_1 \notin B_1$, $t_2 \notin B_2, \ldots,$ and $t_u \notin B_u$. Avoiding these ("finitely" many) points, without loss of generality suppose that $\eta_1 > 0$ is such that $e_{x_{1,\eta}}, \ldots, e_{y_{u,\eta}}, x_1^*, \ldots, x_n^*$ are linearly independent whenever $0 < \eta \leqslant \eta_1$. But then $e_{t_1}, \ldots, e_{t_u}, x_1^*, \ldots, x_n^*$ are linearly dependent if and only if

$$\min_{\substack{\alpha \in R^{n+u} \\ \|\alpha\|=1}} \left\| \sum_{i=1}^{n} \alpha_i x_i^* + \sum_{i=n+1}^{n+u} \alpha_i e_{t_i} \right\| = 0,$$

whence by continuity of the expression (8) in the $2n$-variables $x_{1,\eta}, \ldots, y_{u,\eta}$, the expression (8) attains its minima for some η_2 over the range $0 \leqslant \eta \leqslant \eta_1$. But $0 \leqslant \eta_2 \leqslant \eta_1$ implies the expression (8) is strictly positive. ∎

An application of Lemmas 5 and 8 now yields

COROLLARY 5. *Suppose $e_{t_1}, \ldots, e_{t_u}, x_1^*, \ldots, x_n^*$ are linearly independent functionals on $C[a, b]$. Suppose $x_{i,\eta}, y_{i,\eta}$ are as above. Suppose $m \in M$, $\| m \| < 1$. Then there exists a positive constant η'' such that we may choose the $\tau > 0$ independent of $0 < \eta \leqslant \eta''$ so that given $0 < \eta \leqslant \eta''$ and $d \in \mathbb{R}^n$, $\| d - c \| < \tau$ there is an $r = r(\eta, d) \in \Pi$ for which*

(i) $x_i^* r = x_i^* m + d_i \quad (i = 1, \ldots, n)$,
(ii) $r(x_{i,\eta}) = m(x_{i,\eta}) \quad (i = 1, \ldots, u)$,
(iii) $r(y_{i,\eta}) = m(y_{i,\eta}) \quad (i = 1, \ldots, u)$, and
(iv) $\| r \| < 1$,

where $c = (c_1, \ldots, c_n)$, $c_i = x_i^* m$ $(i = 1, \ldots, n)$.

THEOREM 1. *Suppose $f \in C[a, b] \backslash \Pi$, $\| f \| = 1$. Suppose $x_1^*, \ldots, x_n^*$ is a SAIN sequence in $C[a, b]$ which is nonextremal on $C[a, b] \backslash \Pi$. If $e_{t_1}, \ldots, e_{t_u}$, $x_1^*, \ldots, x_n^*$ is a SAIN sequence in $C[a, b]$, then given $\epsilon > 0$ arbitrary there is a $p \in \Pi$ for which*

(i) $x_i^* p = x_i^* f \quad (i = 1, \ldots, n)$,
(ii) $p(t_j) = f(t_j) \quad (j = 1, \ldots, u)$,
(iii) $\| p \| = \| f \|$, and
(iv) $\| f - p \| < \epsilon$.

Proof. Let $w_i^* = x_i^* \circ \chi_{[a,b] \backslash \{t_1, \ldots, t_u\}}$. By Corollary 4, $w_1^*, \ldots, w_n^*$ is a SAIN sequence on $C[a, b]$. By Lemma 7, $w_1^*, \ldots, w_n^*$ is nonextremal for $C[a, b] \backslash \Pi$. By Lemma 1, given $\epsilon > 0$ there is an $m \in \Pi$ for which

$$w_i^* m = w_i^* f, \qquad (i = 1, \ldots, n)$$
$$\| m \| < \| f \| \qquad \text{and} \qquad \| f - m \| < \epsilon/4.$$

For $0 < \xi < \xi_0 = \min[|t_i - t_j|; i, j = 1,..., u, i \neq j]/2$, consider functions g_ξ defined as

$$\begin{aligned} g_\xi(x) &= \xi^{-1}(f(t_i) - m(t_i - \xi))(x - t_i) + f(t_i), && \text{if} \quad x \in [t_i - \xi, t_i] \\ &= \xi^{-1}(m(t_i + \xi) - f(t_i))(x - t_i) + f(t_i), && \text{if} \quad x \in [t_i, t_i + \xi] \\ &= m(x), && \text{otherwise.} \end{aligned}$$

For $0 < \xi < \xi_0$ likewise define $T_\xi = \{x \in [a, b]; |x - t| \leqslant \xi \text{ for some } t \in T\}$, T being $\{t_1, ..., t_u\}$. Also let $v_{i,\xi}^*$, $y_{i,\xi}^*$ denote the linear functionals

$$v_{i,\xi}^* = x_i^* \circ \chi_{[a,b]\setminus T_\xi},$$

$$y_{i,\xi}^* = w_i^* - v_{i,\xi}^* = x_i^* \circ \chi_{T_\xi \setminus T}.$$

Since the measure of $T_\xi \setminus T$ tends to zero as $\xi \to 0^+$, $\| y_{i,\xi}^* \| \to 0$, and hence $v_{i,\xi}^* \to w_i^*$, as $\xi \to 0^+$. Since g_ξ and m differ at most on T_ξ only,

$$\begin{aligned} w_i^* g_\xi &= v_{i,\xi}^* g_\xi + y_{i,\xi}^* g_\xi \\ &= v_{i,\xi}^* m + y_{i,\xi}^* g_\xi \to w_i^* m, \qquad \text{as} \quad \xi \to 0. \end{aligned}$$

By Corollary 2, let $x_{i,\eta}$, $y_{i,\eta}$ be sequences of points in $[a, b]$ satisfying (6), with (v) replaced by

$$\text{(v')} \quad e_{x_{1,\eta}}, ..., e_{y_{u,\eta}}, e_{t_1}, ..., e_{t_u}, x_1^*, ..., x_n^* \tag{9}$$

is a SAIN sequence on $C[a, b]$.

We now establish several technical results before completing the proof of Theorem 1.

LEMMA 9. *Under the notation above, suppose* $m' \in \Pi$, $\| m' \| < 1$. *Then there exists an* $\eta_0 > 0$ *such that, for* $0 < \eta \leqslant \eta_0$, *there exists a* $q = q(\eta)$, $q \in \Pi$, *for which*

$$\begin{aligned} &\text{(i)} \quad v_{i,\eta}^* q = w_i^* m' && (i = 1,..., n), \\ &\text{(ii)} \quad q(x_{i,\eta}) = m'(x_{i,\eta}) && (i = 1,..., u), \\ &\text{(iii)} \quad q(y_{i,\eta}) = m'(y_{i,\eta}) && (i = 1,..., u), \quad \textit{and} \\ &\text{(iv)} \quad \| q \| < 1. \end{aligned} \tag{10}$$

Proof. For $\eta > 0$, let $\tau = \tau(\eta) > 0$ be the positive constant τ given by Corollary 3 for the linear functionals listed in (9). Set $\alpha = \tau(1 - \| m' \|)/2(n + 2u)$ and let $h = h_{i_1, i_2 \cdots i_l} \in \mathbb{R}^n$ be an n-vector whose i_1st, i_2nd,..., i_lth components are $-\alpha$, and whose other components are α.

Let $h_{(i)}$ denote the ith component of h. Letting h also stand for the $(n + 2u)$-vector whose first n components are the components of h and whose last $2u$ components are zero, $\| h \| < \tau(1 - \| m' \|)$, so by Corollary 3 there is an $s = s_{i, i_2 \cdots i_l} \in \Pi$ for which

(i) $w_i{}^* s = h_{(i)} \quad (i = 1,..., n)$,

(ii) $s(t_i) = 0 \quad (i = 1,..., u)$,

(iii) $s(x_{i,\eta}) = 0 \quad (i = 1,..., u)$,

(iv) $s(y_{i,\eta}) = 0 \quad (i = 1,..., u)$, and

(v) $\| s \| < 1 - \| m' \|$.

Choose $\eta_0 > 0$ so that $\| y_{i,\eta}^* \| < \alpha/n^2$ holds whenever $0 < \eta \leqslant \eta_0$, $i = 1,..., n$. Note that $\alpha(\eta)$ is bounded as $\eta \to 0^+$. Then for every $0 < \eta \leqslant \eta_0$ and $i_1 ,..., i_l$,

$$\begin{aligned} \operatorname{sgn} v_{i,\eta}^* s_{i_1, i_2 \cdots i_l} = \operatorname{sgn} w_i{}^* s_{i_1 i_2 \cdots i_l} &= -1, \qquad \text{if} \quad i \in \{i_1 ,..., i_l\}, \\ &= 1, \qquad \text{otherwise}. \end{aligned}$$

We may now choose $0 < \lambda_1 < 1$ so that

$$v_{i_1,\eta}^* (\lambda_1 s_{i_1, i_2 \cdots i_l} + (1 - \lambda_1)\, s_{i_2 \cdots i_l}) = y_{i_1,\eta}^* m'.$$

Setting $g = g_{i_1; i_2 \cdots i_l} = \lambda_1 s_{i_1, i_2 \cdots i_l} + (1 - \lambda_1)\, s_{i_2 \cdots i_l}$, observe that $v_{i_1,\eta}^* g = y_{i_1,\eta}^* m'$, $g \in \Pi$, $g(t_i) = 0$ $(i = 1,..., u)$, $g(x_{i,\eta}) = 0$ $(i = 1,..., n)$, $g(y_{i,\eta}) = 0$, $(i = 1,..., n)$, $\| g \| < 1 - \| m' \|$, and

$$\begin{aligned} w_i{}^* g = h_{(i)} &= -\alpha, \qquad \text{if} \quad i \in \{i_2 ,..., i_l\} \\ &= \alpha, \qquad \text{otherwise} \quad (i \neq i_1). \end{aligned}$$

In particular, for $i \neq i_1$,

$$\begin{aligned} \operatorname{sgn} v_{i,\eta}^* g_{i_1, i_2 \cdots i_l} = \operatorname{sgn} w_i{}^* g_{i_1, i_2 \cdots i_l} &= -1, \qquad \text{if} \quad i \in \{i_2 \cdots i_l\}, \\ &= 1, \qquad \text{otherwise}. \end{aligned}$$

Suppose now that $g = g_{i_1 \cdots i_\psi ; i_{\psi+1} \cdots i_l} \in \Pi$ have been found so that

$$\begin{aligned} v_{i,\eta}^* g &= y_{i,\eta}^* m', \qquad (i \in \{i_1 ,..., i_\psi\}), \\ w_i{}^* g &= -\alpha, \qquad \text{if} \quad i \in \{i_{\psi+1} ,..., i_l\} \\ &= \alpha, \qquad \text{otherwise} \quad (i \neq i_1 ,..., i_\psi), \end{aligned} \tag{11}$$

and

$$\| g \| < 1 - \| m' \|.$$

Since $\| y^*_{i,\eta} \| < \alpha/n^2$, it follows from (11) that, for $i \neq i_1, ..., i_\psi$,

$$\operatorname{sgn} v^*_{i,\eta} g = \operatorname{sgn} w_i{}^* g = -1, \quad \text{if} \quad i \in \{i_{\psi+1}, ..., i_l\}$$
$$= 1, \quad \text{otherwise.}$$

Thus we may find a $0 < \lambda_{\psi+1} < 1$ so that

$$v^*_{i_{\psi+1}}(\lambda_{\psi+1} g_{i_1 \cdots i_\psi; i_{\psi+1} i_{\psi+2} \cdots i_l} + (1 - \lambda_{\psi+1})\, g_{i_1 \cdots i_\psi; i_{\psi+2} \cdots i_l}) = y^*_{i_{\psi+1}} m'.$$

Setting

$$g_{i_1 \cdots i_\psi i_{\psi+1}; i_{\psi+2} \cdots i_l} = \lambda_{\psi+1} g_{i_1 \cdots i_\psi; i_{\psi+1} \cdots i_l} + (1 - \lambda_{\psi+1})\, g_{i_1 \cdots i_\psi; i_{\psi+2} \cdots i_l}$$

we have that

$$g = g_{i_1 \cdots i_{\psi+1}; i_{\psi+2} \cdots i_l} \in \Pi,$$

$$v^*_{i,\eta} g = y^*_{i,\eta} m' \quad (i \in \{i_1, ..., i_{\psi+1}\}), \qquad g(t_i) = 0, \quad (i = 1, ..., u),$$

$$g(x_{i,\eta}) = 0 \quad (i = 1, ..., u), \qquad g(y_{i,\eta}) = 0, \quad (i = 1, ..., u),$$

$$\| g \| < 1 - \| m' \|,$$

and

$$w_i{}^* g = h_{(i)} = -\alpha, \quad \text{if} \quad i \in \{i_{\psi+2}, ..., i_l\}$$
$$= \alpha, \quad \text{otherwise} \quad (i \neq i_1, ..., i_{\psi+1}).$$

By construction there is therefore a $g = g_{1,2,...,n} \in \Pi$ for which $v^*_{i,\eta} g = y^*_{i,\eta} m'$ $(i = 1, ..., n)$, $g(t_i) = 0$ $(i = 1, ..., u)$, $g(x_{i,\eta}) = 0$ $(i = 1, ..., u)$, $g(y_{i,\eta}) = 0$ $(i = 1, ..., u)$, and $\| g \| < 1 - \| m' \|$. Setting $q = m' + g$, $q \in \Pi$, $q(x_{i,\eta}) = m'(x_{i,\eta})$ $(i = 1, ..., u)$, $q(y_{i,\eta}) = m'(y_{i,\eta})$ $(i = 1, ..., u)$, $\| q \| < 1$, and

$$v^*_{i,\eta} q = v^*_{i,\eta} m' + v^*_{i,\eta} g = v^*_{i,\eta} m' + y^*_{i,\eta} m' = w_i{}^* m' \qquad (i = 1, ..., n). \quad \blacksquare$$

LEMMA 10. *Under the notation above, suppose $m' \in \Pi$, $\| m' \| < 1$. Then there exist positive η_1 and τ' such that for any $0 < \eta \leqslant \eta_1$ and $\| d \| < \tau\sigma$, $d \in R^n$, there is an $m'' \in \Pi$ for which*

(i) $v_{i,\eta} m'' = w_i{}^* m' + d_i \quad (i = 1, ..., n)$,

(ii) $m''(x_{i,\eta}) = 0 \quad (i = 1, ..., u)$,

(iii) $m''(y_{i,\eta}) = 0 \quad (i = 1, ..., u)$, *and*

(iv) $\| m'' \| < \sigma$.

Proof. For $0 < \eta \leqslant \eta_0$, let $\tau = \tau(\eta) > 0$ be the positive constant given by Corollary 3 for the linear functionals listed in (9). Since $\mathbb{R}^n$ is finite dimensional, suppose $d^{(1)},..., d^{(n)}$ is a basis of $\mathbb{R}^n$, and that $\| d^{(j)} \| < \tau$ for each $j = 1,..., n$. By Corollary 3 there is then an $m_j = m(d^{(j)}) \in \Pi$ such that $v_i{}^*m_j = d_i^{(j)}$ $(i = 1,..., n)$, $m_j(t_i) = 0$ $(i = 1,..., u)$, and $\| m_j \| < 1$. By Lemma 9, there are positive constants $\eta_j = \eta_0(d^{(j)})$ such that given $0 < \eta \leqslant \eta_j$ there is a $q_j = q_j(\eta) \in \Pi$ for which (10) holds, with $v_{i,\eta}^* q_j = w_i{}^*m' + d_i^{(j)}$ $(i = 1,..., n)$. Set $\gamma = \min[\eta_j\, ; j = 0, 1,..., n]$. Then if $0 < \eta \leqslant \gamma$, $\xi \in \mathbb{R}^n$, $\| \xi \| \leqslant 1$, $d \in \mathbb{R}^n$, $d = \sum_{j=1}^n \xi_j d^{(j)}$ and $\| d \| < \sigma\tau/n$, there is a $q \in \Pi$ for which $v_{i,\eta}^* q = w_i{}^*m' + d_i$ $(i = 1,..., n)$, $q(x_{i,\eta}) = m'(x_{i,\eta})$ $(i = 1,..., u)$, $q(y_{i,\eta}) = m'(y_{i,\eta})$ $(i = 1,..., u)$, and $\| q \| < \sigma$. But $\{d \in \mathbb{R}^n;$ $d = \sum_{j=1}^n \xi_j d^{(j)}$ for some $\xi \in \mathbb{R}^n$, $\| \xi \| \leqslant 1$, $\| d \| < \tau/n\}$ contains a nonempty open ball about the origin. Set τ' equal to its radius. We need merely show that we could in fact choose τ' independent of η for $0 < \eta < \eta_1$, for some $\eta_1 \leqslant \gamma$. But by Corollary 5 there is an $\beta > 0$ for which it is possible to choose the τ given by Corollary 3 independently of η in the range $0 < \eta \leqslant \beta$. Setting $\eta_1 = \min[\gamma, \beta]$ we are done. ∎

COROLLARY 6. *Under the notation above, there exist positive constants* η_0, τ *such that* $0 < \eta < \eta_0$ *and* $\| d \| < \tau\sigma/2$, $d \in \mathbb{R}^n$ *implies there is an* $m' \in \Pi$ *for which*

(i) $v_{i,\eta}^* m' = d_i$ $\quad (i = 1,..., n)$,

(ii) $m'(x_{i,\eta}) = 0$ $\quad (i = 1,..., u)$,

(iii) $m'(y_{i,\eta}) = 0$ $\quad (i = 1,..., u)$, *and*

(iv) $\| m' \| < \sigma$.

Proof. Apply Lemma 10 twice, once to $d^{(1)} = 0$ and once to $d^{(2)} = d$, getting an m_1'' and m_2'', respectively. Set $m' = m_2'' - m_1''$. ∎

Returning to the proof of Theorem 1, pick a positive $\xi_1 < \eta_0$ for which $\sum_{j=1}^n \| y_{i,\xi}^* \| < \tau\epsilon(1 - \| m \|)/4$ whenever $0 < \xi < \xi_1$. Choose a $\xi_2 > 0$ so that $T_{\xi_2} \subseteq \{x \in [a, b];\ | g_\xi(x) - m(x)| < \epsilon/4\}$. Set $\xi_0 = \min[\xi_1, \xi_2]$. Then for $0 < \xi < \xi_0$, set $d_i = -y_{i,\xi}^* g_\xi$. By Corollary 6, for any $\xi_0 < \eta < \eta_0$ there is an $m' \in \Pi$ for which

(i) $v_{i,\eta}^* m' = -y_{i,\xi}^*(g_\xi - m)$ $\quad (i = 1,..., n)$,

(ii) $m'(x_{i,\eta}) = 0$ $\quad (i = 1,..., u)$,

(iii) $m'(y_{i,\eta}) = 0$ $\quad (i = 1,..., u)$, and

(iv) $\| m' \| < (1 - \| m \|)\epsilon/2$.

Define a new sequence of functions h_ξ as

$$\begin{aligned} h_\xi(x) &= g_\xi(x) + m'(x), && \text{if } x \in [a, b]\backslash T_\eta \\ &= g_\xi(x), && \text{otherwise.} \end{aligned}$$

Then $h_\xi \in C[a, b]$, $h_\xi(t_i) = f(t_i)$, $(i = 1,..., u)$, $\|f - h_\xi\| \leqslant \|f - g_\xi\| + \|m'\| \leqslant \epsilon/2$, and

$$\begin{aligned} w_i{}^*h_\xi &= v_{i,\eta}^*(g_\xi + m') + y_{i,\eta}^* g_\xi \\ &= w_i{}^*g_\xi - y_{i,\xi}^* g_\xi + y_{i,\xi}^* m \\ &= v_{i,\xi}^* m + y_{i,\xi}^* m \\ &= w_i{}^*m \\ &= w_i{}^*f, \qquad (i = 1,..., n). \end{aligned}$$

In particular, then, $x_i{}^* = w_i{}^* + x_i{}^* \circ \chi_T$ implies $x_i{}^*h_\xi = x_i{}^*f\,(i = 1,..., n)$.

Finally $\|h_\xi\| \leqslant \|f\| = 1$, for $h_\xi = g_\xi$ on T_η, and $\|g_\xi\| \leqslant 1$, while on $[a, b]\backslash T_\eta$ $g_\xi = m(x)$, and $|m(x)| \leqslant \|m\|$, whence $|h_\xi(x)| \leqslant \|m\| + (1 - \|m\|)/2 < 1$ whenever $x \in [a, b]\backslash T_\eta$. By Proposition C, there is a $p \in \Pi$ for which

(i) $x_i{}^*p = x_i{}^*h_\xi \quad (i = 1,..., n)$,

(ii) $p(t_j) = h_\xi(t_j) \quad (j = 1,..., u)$,

(iii) $\|p\| = \|h_\xi\|$, and

(iv) $\|h_\xi - p\| < \epsilon/2$.

Since $x_i{}^*h_\xi = x_i{}^*f\,(i = 1,..., n)$, $h_\xi(t_j) = f(t_j)\,(j = 1,..., u)$, $\|h_\xi\| \leqslant \|f\|$ and $\|f - h_\xi\| \leqslant \epsilon/2$, done. ∎

3. Necessity

Suppose $z_1'^*,..., z_n'^*$ is an arbitrary SAIN sequence on $C[a, b]$, and set $Z = \langle z_1'^*,..., z_n'^*\rangle$. We find a different basis of Z as follows: if $z_1'^*,..., z_n'^*$ is nonextremal for $C[a, b]\backslash\Pi$, let $z_1{}^*,..., z_n{}^*$ be n arbitrary linearly independent elements of Z. Otherwise, let $z_1{}^*$ be a nonzero linear functional in Z which is extremal with respect to $C[a, b]\backslash\Pi$. If $Z = \langle z_1{}^*\rangle \oplus Z_1$, and the elements of Z_1 are all nonextremal for $C[a, b]\backslash\Pi$, let $z_2{}^*,..., z_n{}^*$ be arbitrary linearly independent elements of Z_1. Otherwise let $z_2{}^*$ be a nonzero functional in Z_1 which is extremal on $C[a, b]\backslash\Pi$. Continuing in this manner we may find a maximal sequence of linearly independent functionals $z_1{}^*,..., z_r{}^*$, each of which is extremal on $C[a, b]\backslash\Pi$. We then let $z_{r+1}^*,..., z_n{}^*$ be arbitrary functionals in Z such that $z_1{}^*,..., z_n{}^*$ are linearly independent. By Lemma 2, a SAIN functional is extremal on $C[a, b]\backslash\Pi$ if and only if it is finitely purely atomic, while $z_{r+1}^*,..., z_n{}^*$ forms a SAIN sequence which is nonextremal on $C[a, b]\backslash\Pi$.

In the previous section we considered the case when the finitely purely atomic functionals $z_1^*,..., z_r^*$ were point evaluations (one atom only) and showed that one had property SAIN holding for such SAIN sequences. We now handle the case when the $z_1^*,..., z_r^*$ may have more than one atom apiece.

Observe first that Theorem 1 implies that the triple $(C[a, b], \Pi, \{z_1^*,..., z_r^*, z_{r+1}^*,..., z_n^*\})$ has property SAIN whenever the (finitely many) atoms of $z_1^*,..., z_r^*$, call them $t_1,..., t_u$, are such that the sequence $e_{t_1},..., e_{t_u}, z_{r+1}^*,..., z_n^*$ is a SAIN sequence. For if $g \in C[a, b]$ and $\epsilon > 0$ is arbitrary, if there must exist a $p \in \Pi$ for which

(i) $p(t_i) = g(t_i)$ $(i = 1,..., u)$,

(ii) $z_j^*p = z_j^*g$ $(j = r + 1,..., n)$,

(iii) $\| p \| = \| g \|$, and

(iv) $\| g - p \| < \epsilon$,

then

$$z_j^* = \sum_{i=1}^{u} \xi_{ij} e_{t_i}$$

implies

(v) $z_j^*p = z_j^*g$ $(j = 1,..., r)$

also. Furthermore Lemma 3 and Corollary 2 show that relatively few points of $[a, b]$ can be such that $e_{t_1},..., e_{t_u}, z_{r+1}^*,..., z_n^*$ is not a SAIN sequence. Thus in a certain sense, the triple $(C[a, b], \Pi, \{z_1^*,..., z_n^*\})$ will have properly SAIN holding at least for almost all SAIN sequences.

LEMMA 11. *Suppose $x_1^*, x_2^*,..., x_n^*$ is a SAIN sequence on $C[a, b]$, x_1^* purely finitely atomic and $x_2^*,..., x_n^*$ nonextremal on $C[a, b]\backslash\Pi$. If $f \in C[a, b]$ is such that $| x_1^*f | < \| x_1^* \|$, then given $\epsilon > 0$ there is a $p \in \Pi$ for which*

(i) $x_i^*p = x_i^*f$ $(i = 1,..., n)$

(ii) $\| p \| = \| f \|$, *and*

(iii) $\| f - p \| < \epsilon$.

Proof. If $f \in \Pi$, trivial. Thus suppose $f \in C[a, b]\backslash\Pi$, $\| f \| = 1$, and $x^* \in \langle x_1^*,..., x_n^*\rangle\backslash\{0\}$. If $x^* \in \langle x_2^*,..., x_n^*\rangle\backslash\{0\}$, by hypothesis $| x^*f | < \| x^* \|$. Thus suppose $x^* = \xi_1 x_1^* + \sum_{j=2}^{n} \xi_j x_j^*$, with $\xi_1 \neq 0$. Since x^* is a SAIN functional, it is either purely finitely atomic, in $\pm\mathscr{P}$ with support $[a, b]$, or else nonextremal on $C[a, b]$. But x^* finitely purely atomic implies $\sum_{j=2}^{n} \xi_j x_j^*$ is also (or else is the zero functional). Since $\sum_{j=2}^{n} \xi_j x_j^*$ is nonextremal on

$C[a,b]\backslash\Pi$, it is not purely finitely atomic. If $x^* \in \pm\mathscr{P}$ with support $x^* = [a,b]$, $|x^*f| = \|x^*\|$ if and only if $f = \pm\mathbb{1} \in \Pi$. Thus necessarily x^* must be nonextremal on $C[a,b]$, and hence $|x^*f| < \|x^*\|$. If $x^* \in \langle x_1^*\rangle$, $|x_1^*f| < \|x_1^*f\|$ by hypothesis. Since $x^* \in \langle x_1^*,..., x_n^*\rangle\backslash\{0\}$ was arbitrary, $x_1^*,..., x_n^*$ is nonextremal for f, whence by Lemma 1 the conclusion follows. ∎

LEMMA 12. *Suppose $x_1^*,..., x_n^*$ is a SAIN sequence on $C[a,b]$, x_1^* finitely purely atomic and $x_2^*,..., x_n^*$ nonextremal on $C[a,b]\backslash\Pi$. Suppose $T = \{t_1,..., t_u\}$ is the set of atoms of x_1^*, and $w_j^* = x_j^* \circ \chi_{[a,b]\backslash T}$ $(j = 2,..., n)$. If $w_2^*,..., w_n^*$ is nonextremal on $C[a,b]\backslash\Pi$, then given $\epsilon > 0$ and $f \in C[a,b]$ arbitrary, there is a $p \in \Pi$ for which*

(i) $x_i^*p = x_i^*f$ $(i = 1,..., n)$,
(ii) $\|p\| = \|f\|$, *and*
(iii) $\|f - p\| < \epsilon$.

Proof. Suppose that $\|f\| = 1$. By Lemma 1 there is an $m \in \Pi$ for which

(i) $w_j^*m = w_j^*f$ $(j = 2,..., n)$,
(ii) $\|m\| < \|f\|$, and
(iii) $\|f - m\| < \epsilon/4$.

Observing the proof of Theorem 1 closely, the fact that $e_{t_1},..., e_{t_u}$, $x_2^*,..., x_n^*$ was a SAIN sequence was critical only in obtaining such an $m \in \Pi$ as above (the choice of $x_{i,n}$, $y_{i,n}$ using Corollary 2 so that the sequence in (9) is a SAIN sequence may be modified by employing Lemma 4 in place of Corollary 2 and getting the sequence of linear functions in (9) to be linear independent, and the linear independence of the sequence (9) was all that was really used in the balance of the proof). Hence repeating the proof of Theorem 1 yields the desired conclusion. ∎

Lemmas 11 and 12 give sufficient conditions in order that a SAIN sequence $x_1^*, x_2^*,..., x_n^*$ with x_1^* finitely purely atomic and $x_2^*,..., x_n^*$ nonextremal on $C[a,b]\backslash\Pi$ be such that the triple $(C[a,b], \Pi, \{x_1^*,..., x_n^*\})$ have property SAIN. We now show that the hypotheses for at least one of Lemmas 11 and 12 must be satisfied for $x_1^*,..., x_n^*$ a SAIN sequence with x_1^* purely finitely atomic and $x_2^*,..., x_n^*$ nonextremal.

LEMMA 13. *Suppose $x_1^*,..., x_n^*$ is a SAIN sequence on $C[a,b]$, x_1^* purely finitely atomic and $x_2^*,..., x_n^*$ nonextremal on $C[a,b]\backslash\Pi$. Suppose $T = \{t_1,..., t_u\} = \operatorname{supp} x_1^*$ and $w_j^* = x_j^* \circ \chi_{[a,b]\backslash T}$ $(j = 2,..., n)$. Then there cannot exist a $g \in C[a,b]\backslash\Pi$ and a $w^* \in \langle w_2^*,..., w_n^*\rangle\backslash\{0\}$ for which both*

(i) $|x_1^*g| = \|x_1^*\|\,\|g\|$, *and*
(ii) $|w^*g| = \|w^*\|\,\|g\|$

unless w^* should also be finitely purely atomic.

Proof. Suppose not and let $x^* \in \langle x_2^*, ..., x_n^*\rangle\backslash\{0\}$ be such that $w^* = x^* \circ \chi_{[a,b]\backslash T}$. Then $x^* = w^* + \sum_{i=1}^{u} \alpha_i e_{t_i}$ for some $\alpha = (\alpha_i) \in \mathbb{R}^u$. By hypothesis (ii) $g \in C[a, b]\backslash \Pi$ must be such that

$$\begin{aligned} g(x) &= 1, && \text{if} \quad x \in \operatorname{supp} w^{+*}, \\ &= -1, && \text{if} \quad x \in \operatorname{supp} w^{-*}. \end{aligned}$$

In particular g continuous requires that $\operatorname{supp} w^{+*} \cap \operatorname{supp} w^{-*} = \varnothing$, and $g \neq \mathbb{1}$ requires then that $\operatorname{supp} w^* \neq [a, b]$. But x^* must be a SAIN functional (since x_1^*, x^* is a SAIN sequence) and so Lemma 2 requires that either x^* is finitely purely atomic, has support $[a, b]$, or else that $\operatorname{supp} x^{+*} \cap \operatorname{supp} x^{-*} \neq \varnothing$. x^* being finitely purely atomic implies w^* is also, and likewise $\operatorname{supp} x^* = [a, b]$ implies $\operatorname{supp} w^* = [a, b]$, neither of which are true. Hence $\operatorname{supp} x^{+*} \cap \operatorname{supp} x^{-*} \neq \varnothing$.

Suppose that $x_1^* = \sum_{i=1}^{u} \beta_i e_{t_i}$ and that $T = \{t_1, ..., t_u\}$ is such that $t_i \in \operatorname{supp} x^{+*}$ $(i = 1, ..., s)$, $t_i \in \operatorname{supp} x^{-*}$ $(i = s + 1, ..., s')$, and $t_i \notin \operatorname{supp} x^*$ $(i = s' + 1, ..., u)$, $0 \leqslant s \leqslant s' \leqslant u$ with either $s \geqslant 1$ or $s' \geqslant s + 1$ (or both). Let $\gamma = \max\{|\alpha_i|/|\beta_i|; i = 1, ..., s'\}$ and consider the functional $z^* = x^* + \gamma x_1^*$. Since hypothesis (i) requires that

$$\begin{aligned} g(t_i) &= 1, && \text{if} \quad 1 \leqslant i \leqslant s \\ &= -1, && \text{if} \quad s + 1 \leqslant i \leqslant s', \end{aligned}$$

we observe that z^* is neither finitely purely atomic, does not have support $[a, b]$, and that $\operatorname{supp} z^{+*} \cap \operatorname{supp} z^{-*} = \varnothing$ (for $z^* = x^* + \gamma x_1^* = w^* + \sum_{i=1}^{s'} (\alpha_i + \gamma\beta_i)\, e_{t_i} + \sum_{i=s'+1}^{u} (\alpha_i + \gamma\beta_i)\, e_{t_i}$ and we observe that $\alpha_i + \gamma\beta_i \geqslant 0$ if $1 \leqslant i \leqslant s$, while $\alpha_i + \gamma\beta_i \leqslant 0$ if $s + 1 \leqslant i \leqslant s'$). Hence z^* is a non-SAIN linear functional, contradicting the assumption that $x_1^*, ..., x_n^*$ is a SAIN sequence on $C[a, b]$. ∎

Remark 2. If $x_1^*, ..., x_r^*$, $x_{r+1}^*, ..., x_n^*$ is a SAIN sequence with $x_1^*, ..., x_r^*$ finitely purely atomic and $x_{r+1}^*, ..., x_n^*$ nonextremal on $C[a, b]\backslash \Pi$, and if $f \in C[a, b]$, $\|f\| = 1$ is such that $|x_i^* f| = \|x_i^*\|$ for each $i = 1, ..., r$, it is possible to replace $x_1^*, ..., x_r^*$ by a z^* such that given a $p \in \Pi$, $\|p\| \leqslant 1$, $x_i^* p = x_i^* f$ $(i = 1, ..., n)$ if and only if $z^* p = z^* f$. Moreover this z^* will be such that $|z^* f| = \|z^*\|$. Thus without loss of generality we may suppose that for a given $f \in C[a, b]$, $\|f\| = 1$, that $|x^* f| < \|x^*\|$ for every $x^* \in \langle x_2^*, ..., x_r^*\rangle\backslash\{0\}$.

Remark 3. Actually it is possible to improve on Remark 2 by following a procedure similar to that in obtaining the $z_1^*, ..., z_n^*$ from the $z_1'^*, ..., z_n'^*$ specified at the beginning of this section. For, if $f \in C[a, b]\backslash \Pi$, $\|f\| = 1$ is arbitrary, let $z_1^* \in X = \langle x_1^*, \ldots, x_n^* \rangle \backslash \{0\}$ be such that $|z_1^* f| = \| z_1^* \|$ (if such exist), $z_2^* \in X$ so that z_1^* and z_2^* are linearly independent and $|z_2^* f| = \| z_2^* \|, \ldots, z_r^* \in X$ so that $z_1^*, \ldots, z_r^*$ are linearly independent and $|z_r^* f| = \| z_r^* \|$. We then find a $0 \leqslant r \leqslant n$ so that the sequence of $z_1^*, \ldots, z_r^*$ is maximal (i.e., so that $z_{r+1}^*, \ldots, z_n^*$ is nonextremal for f if $z_1^*, \ldots, z_r^*, z_{r+1}^*, \ldots, z_n^*$ are linearly independent elements and a basis for X). By Remark 2, it is possible to replace $z_1^*, \ldots, z_r^*$ by an equivalent (sense specified in Remark 2) extremal finitely purely atomic functional z^*, and we find ourselves considering the case dealt with in Lemmas 11–13. Thus given an $f \in C[a, b]$ and $\epsilon > 0$ arbitrary it is possible to conclude whether there exist $p \in \Pi$ for which (i) $x_i^* p = x_i^* f$ $(i = 1, \ldots, n)$, (ii) $\| p \| = \| f \|$, and (iii) $\| f - p \| < \epsilon$ on the basis of the linear functionals $x_1^*, \ldots, x_n^*$ alone, a sufficient condition being that $x_1^*, \ldots, x_n^*$ form a SAIN sequence on $C[a, b]$. In other words we have shown that:

THEOREM 2. *The triple $(C[a, b], \Pi, \{x_1^*, \ldots, x_n^*\})$ has property SAIN if and only if the sequence $x_1^*, \ldots, x_n^*$ is a SAIN sequence.*

4. Some Examples

EXAMPLE 4.1.

$$x_1^* = e_{1/8},$$

$$x_2^* = \int_0^{1/2} \cdot \, dx - e_{1/4},$$

$$x_3^* = 3e_{1/4} - 2e_{1/8} - \int_0^{3/4} \cdot \, dx.$$

Here x_1^*, x_2^*, x_3^* is not a SAIN sequence on $C[0, 1]$, since $2x_1^* + x_2^* + x_3^* = 2e_{1/4} - \int_{1/2}^{3/4} \cdot \, dx$ is not a SAIN functional. Hence $(C[0, 1], \Pi, \{x_1^*, x_2^*, x_3^*\})$ does not have property SAIN.

EXAMPLE 4.2.

$$x_1^* = \int_0^1 \cdot \, dx - e_{1/4} - e_{5/8},$$

$$x_2^* = \int_0^{1/2} \cdot \, dx - e_{1/3}.$$

Here x_1^*, x_2^* is a SAIN sequence which is nonextremal on $C[0, 1]$. Hence property SAIN holds.

EXAMPLE 4.3.

$$x_1^* = e_{1/4} + e_{1/2},$$

$$x_2^* = \int_{3/4}^{1} \cdot dx - \sum_{i=1}^{\infty} 2^{-i} e_{s_i}, \ \{s_i\}_{i=1}^{\infty}$$

rationals in [3/4, 1].

Here x_1^*, x_2^* is a SAIN sequence, x_1^* is finitely purely atomic with atoms 1/4, 1/2, and $e_{1/4}$, $e_{1/2}$, x_2^* is also a SAIN sequence. Since x_2^* is nonextremal, property SAIN holds.

EXAMPLE 4.4.

$$x_1^* = e_{1/2} + e_{3/4} - (1/2)\, e_{1/4},$$

$$x_2^* = e_{1/2} + e_{3/4} + \int_0^{1/2} \cdot dx - e_{1/4}.$$

Here x_1^*, x_2^* is a SAIN sequence, x_1^* is finitely purely atomic with atoms 1/4, 1/2, 3/4; x_2^* is nonextremal, and $e_{1/4}$, $e_{1/2}$, $e_{3/4}$, x_2^* is not a SAIN sequence ($x_2^* - e_{1/2} - e_{3/4} + e_{1/4} = \int_0^{1/2} \cdot dx$ is not a SAIN functional). On the other hand, if $g \in C[a, b]$, $\| g \| = 1$ is such that $| x_1^* g | = \| x_1^* \|$, then (wlog) $g(1/2) = g(3/4) = 1$, $g(1/4) = -1$, whence $| w^* g | < \| w^* \|$, where $w^* = \int_0^{1/2} \cdot dx$. Hence property SAIN holds.

5. GENERALIZATIONS?

The characterizations obtained in Sections 2 and 3 and summarized as Theorem 2 in Section 3 assume that the dense subspace M of $C[a, b]$ is the polynomials Π. Actually this is somewhat stronger an assumption than necessary. For example, if M should be any dense subalgebra (containing the constants) of the polynomials Π, identically the same characterizations as given in Theorem 2 hold. In fact, if M is any dense subalgebra of $C[a, b]$ for which the SAIN functionals with respect to M are the same as those with respect to Π, the same conclusion may be true (is true if in addition any $m \in M$ can attain its norm only finitely often).

For more general dense subspaces of $C[a, b]$, the characterizations analogous to Theorem 2 appear to vary to some extent. One underlying reason is that the SAIN functionals with respect to different dense subspaces can differ to some extent (e.g., consider the SAIN functionals of a dense subalgebra not containing the constant functions. Type ii (positive or negative linear functionals with support $[a, b]$) are no longer SAIN functionals then).

A special example of dense subalgebras M that we can immediately give characterizations analogous to Theorem 2 are the subspaces of the form:

$$M = M(D) = \{f \in C[a, b]; f \text{ coincides with a polynomial in each connected component of } [a, b] \backslash D\}, \qquad (12)$$

where D is a finite union of subintervals of $[a, b]$. Moreover, considering such subspaces M as in (12) is equivalent to considering the underlying function space to be $C(T)$, where T is a disjoint union of compact intervals and M is the direct sum of the spaces of polynomials on each component of T.

For all these special cases, the corresponding characterizations seem to possess underlying similarities to those when $M = \Pi$ (in particular, every SAIN sequence seems to be such that property SAIN holds). Perhaps a closer examination and determination of these similarities would allow one to produce characterizations of property SAIN for more dense subalgebras (and subspaces).

It should be noted that although the use of an interval was fundamental to the proof, the type of characterizations obtained ought also to be valued for more general function spaces $C(T)$, T not one-dimensional.

References

1. F. Deutsch and P. D. Morris, On simultaneous approximation and interpolation which preserves the norm, *Bull. Amer. Math. Soc.* **75** (1969), 812–814.
2. F. Deutsch and P. D. Morris, On simultaneous approximation and interpolation which preserves the norm, *J. Approximation Theory* **2** (1969), 355–373.
3. F. Deutsch and P. D. Morris, Simultaneous approximation and interpolation with preservation of norm, *in* "Approximation Theory," (A. Talbot, Ed.), pp. 309–313. Academic Press, New York, 1970.
4. Dunford and Schwartz, "Linear Operators. Part I," Interscience, New York, 1958.
5. R. Holmes and J. Lambert, A geometrical approach to property (SAIN), *J. Approximation Theory* **7** (1973), 132–142.
6. D. J. Johnson, Jackson type theorems for approximation with side conditions, *J. Approximation Theory* **12** (1974), 213–229.
7. D. J. Johnson, One-sided approximation with side conditions, to appear.
8. J. Lambert, Simultaneous approximation and interpolation in l_1, *Proc. Amer. Math. Soc.* **32** (1972), 150–152.
9. J. Lambert, Simultaneous approximation and interpolation in L_1 and $C(T)$, *Pacific J. Math.* **45** (1973), 293–296.
10. H. W. McLaughlin and P. M. Zaretzki, Simultaneous approximation and interpolation with norm preservation, *J. Approximation Theory* **4** (1971), 54–58.
11. H. L. Royden, "Real Analysis," Macmillian, New York, 1963.
12. V. A. Šmatkov, On simultaneous approximation and interpolation in Banach spaces, *Dokl. Akad. Nauk Armyanskoi SSR* **LIII** (1971), 65–70.
13. H. Yamabe, On an extension of the Helly's theorem, *Osaka Math. J.* **2** (1950), 15–17.